The Vasicek Interest Rate Process
Part I - The Stochastic Short Rate

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The Vasicek interest rate model is a mathematical model that describes the evolution of the short rate of interest over time. The short rate is the annualized interest rate at which an entity can borrow money for an infinitesimally short period of time. Vasicek models the short rate as a Ornstein-Uhlenbeck process. An Ornstein-Uhlenbeck process is a mean-reverting process where the short rate is allowed to incorporate random shocks but is pulled back to it's long-term mean whenever it moves away from it. Interest rates exhibit mean reversion, which is the tendency for a stochastic process to return over time to a long-term mean. Vasicek's stochastic differential equation that defines the change in the short rate r_u over the infinitesimally small time interval $[u, u + \delta u]$ is...

$$\delta r_u = \lambda \left(r_\infty - r_u \right) \delta u + \sigma \, \delta W_u \tag{1}$$

When the short rate moves below its long-term mean r_∞ the short rate drift becomes positive and the short rate is pulled upward. When the short rate moves above its long-term mean the short rate drift becomes negative and the short rate is pulled downward. The speed at which the drift is pulled upward of downward is given by the positive valued parameter λ, which measures the speed of mean reversion. The greater the speed the faster the process reverts toward the long-term mean. Random shocks are introduced via the variables σ, which is the annualized short rate volatility, and δW_u, which is the change in the driving Brownian motion over the infinitesimally short time interval $[u, u + \delta u]$.

We will define the short rate at time t to be the short rate at time s (known) plus the sum of the changes in the short rate over the time period $[s, t]$ (random). Using Equation (1) above the equation for the random short rate at time t as a function of the known short rate at time s is...

$$r_t = r_s + \int_s^t \delta r_u \text{ ...where... } t > s \tag{2}$$

In this white paper we will develop the mathematics of the Vasicek short rate stochastic process. To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

We are tasked with pricing a zero coupon bond and for that task we need an expected short rate curve. Our go-forward interest rate assumptions are as follows...

Description	Symbol	Value
Current short rate	r_s	0.04
Long-term short rate mean	r_∞	0.09
Annualized short rate volatility	σ	0.03
Mean reversion rate	λ	0.35

Question 1: Graph the short rate curve (mean and variance) over the time interval $[0, 10]$.

Question 2: What is the short rate mean and variance at the end of years 1 and 3?

Question 3: What is the correlation between the random short rates at the end of years 1 and 3?

Question 4: What is the probability that the random short rate at the end of year 3 will be negative?

Short Rate Equation

We will define the function $f(r_u, u)$ to be a function of time u and the short rate of interest at time u. The equation for the function $f(r_u, u)$ is...

$$f(r_u, u) = \text{Exp}\left\{\lambda\, u\right\}(r_u - r_\infty) \tag{3}$$

The derivatives of Equation (3) above with respect to time u and the short rate of interest at time u are...

$$\frac{\delta f(r_u, u)}{\delta u} = \lambda\,\text{Exp}\left\{\lambda\, u\right\}(r_u - r_\infty) \text{ ...and... } \frac{\delta f(r_u, u)}{\delta r_u} = \text{Exp}\left\{\lambda\, u\right\} \text{ ...and... } \frac{\delta^2 f(r_u, u)}{\delta r_u^2} = 0 \tag{4}$$

Per Ito's Lemma, Equation (3) is once differentiable with respect to time u and twice differentiable with respect to the stochastic short rate r_u. Using a Taylor Series Expansion the equation for the change in $f(r_u, u)$ is...

$$\delta f(r_u, u) = \frac{\delta f(r_u, u)}{\delta u}\,\delta u + \frac{\delta f(r_u, u)}{\delta r_u}\,\delta r_u + \frac{1}{2}\,\frac{\delta^2 f(r_u, u)}{\delta r_u^2}\,\delta r_u^2 \tag{5}$$

Using the derivative calculations in Equation (4) above we can rewrite Equation (5) above as...

$$\delta f(r_u, u) = \lambda\,\text{Exp}\left\{\lambda\, u\right\}(r_u - r_\infty)\,\delta u + \text{Exp}\left\{\lambda\, u\right\}\delta r_u \tag{6}$$

Using Equation (1) above we can rewrite Equation (6) above as...

$$\begin{aligned}
\delta f(r_u, u) &= \lambda\,\text{Exp}\left\{\lambda\, u\right\}(r_u - r_\infty)\,\delta u + \text{Exp}\left\{\lambda\, u\right\}\Big(\lambda\,(r_\infty - r_u)\,\delta u + \sigma\,\delta W_u\Big) \\
&= \lambda\,\text{Exp}\left\{\lambda\, u\right\}(r_u - r_\infty)\,\delta u + \lambda\,(r_\infty - r_u)\,\text{Exp}\left\{\lambda\, u\right\}\delta u + \sigma\,\text{Exp}\left\{\lambda\, u\right\}\delta W_u \\
&= \lambda\,\text{Exp}\left\{\lambda\, u\right\}(r_u - r_\infty)\,\delta u - \lambda\,\text{Exp}\left\{\lambda\, u\right\}(r_u - r_\infty)\,\delta u + \sigma\,\text{Exp}\left\{\lambda\, u\right\}\delta W_u \\
&= \sigma\,\text{Exp}\left\{\lambda\, u\right\}\delta W_u
\end{aligned} \tag{7}$$

After integrating both sides of Equation (7) above we get...

$$\int_s^t \delta f(r_u, u) = \int_s^t \sigma\,\text{Exp}\left\{\lambda\, u\right\}\delta W_u \tag{8}$$

Note that we can rewrite Equation (8) above as...

$$f(r_t, t) - f(r_s, s) = \sigma \int_s^t \text{Exp}\left\{\lambda\, u\right\}\delta W_u \tag{9}$$

Using Equation (9) above and rearranging terms we get...

$$f(r_t, t) = f(r_s, s) + \sigma \int_s^t \text{Exp}\left\{\lambda\, u\right\}\delta W_u \tag{10}$$

Using Equations (3) and (10) above the equation for random short rate at time t as a function of the known short-rate at time s is...

$$\text{Exp}\left\{\lambda\,t\right\}(r_t - r_\infty) = \text{Exp}\left\{\lambda\,s\right\}(r_s - r_\infty) + \sigma \int\limits_s^t \text{Exp}\left\{\lambda\,u\right\}\delta W_u$$

$$\text{Exp}\left\{\lambda\,t\right\} r_t - \text{Exp}\left\{\lambda\,t\right\} r_\infty = \text{Exp}\left\{\lambda\,s\right\} r_s - \text{Exp}\left\{\lambda\,s\right\} r_\infty + \sigma \int\limits_s^t \text{Exp}\left\{\lambda\,u\right\}\delta W_u$$

$$\text{Exp}\left\{\lambda\,t\right\} r_t = \text{Exp}\left\{\lambda\,s\right\} r_s + \text{Exp}\left\{\lambda\,t\right\} r_\infty - \text{Exp}\left\{\lambda\,s\right\} r_\infty + \sigma \int\limits_s^t \text{Exp}\left\{\lambda\,u\right\}\delta W_u$$

$$r_t = \text{Exp}\left\{-\lambda\,t\right\}\left(\text{Exp}\left\{\lambda\,s\right\} r_s + \text{Exp}\left\{\lambda\,t\right\} r_\infty - \text{Exp}\left\{\lambda\,s\right\} r_\infty + \sigma \int\limits_s^t \text{Exp}\left\{\lambda\,u\right\}\delta W_u\right)$$

$$r_t = r_\infty + \text{Exp}\left\{-\lambda\,(t-s)\right\}(r_s - r_\infty) + \text{Exp}\left\{-\lambda\,t\right\}\sigma \int\limits_s^t \text{Exp}\left\{\lambda\,u\right\}\delta W_u \tag{11}$$

Short Rate Distribution

To make the calcuations that follow easier to handle we will make the following function definition...

$$\mu_t = r_\infty + \text{Exp}\left\{-\lambda\,(t-s)\right\}(r_s - r_\infty) \tag{12}$$

Using Equation (12) above we can rewrite short rate Equation (11) above as...

$$r_t = \mu_t + \text{Exp}\left\{-\lambda\,t\right\}\sigma \int\limits_s^t \text{Exp}\left\{\lambda\,u\right\}\delta W_u \tag{13}$$

Using Equation (13) above the equation for the first moment of the distribution of the random short rate at time t given the known short-rate at time s is...

$$\begin{aligned}
\mathbb{E}\Big[r_t\Big] &= \mathbb{E}\Big[\mu_t + \text{Exp}\left\{-\lambda\,t\right\}\sigma \int\limits_s^t \text{Exp}\left\{\lambda\,u\right\}\delta W_u\Big] \\
&= \mu_t + \text{Exp}\left\{-\lambda\,t\right\}\sigma \int\limits_s^t \text{Exp}\left\{\lambda\,u\right\}\mathbb{E}\Big[\delta W_u\Big] \\
&= r_\infty + \text{Exp}\left\{-\lambda\,(t-s)\right\}(r_s - r_\infty)
\end{aligned} \tag{14}$$

The equation for the square of the random short rate at time t as a function of the known short rate at time s (Equation (13) above) is...

$$r_t^2 = \mu_t^2 + 2\,\mu_t\,\text{Exp}\left\{-\lambda\,t\right\}\sigma \int\limits_s^t \text{Exp}\left\{\lambda\,u\right\}\delta W_u + \text{Exp}\left\{-2\,\lambda\,t\right\}\sigma^2 \int\limits_s^t \int\limits_s^t \text{Exp}\left\{\lambda\,(u+v)\right\}\delta W_u\,\delta W_v \tag{15}$$

Using Equation (15) above and Appendix Equations (38) and (45) below the equation for the second moment of the distribution of the random short rate is...

$$\begin{aligned}\mathbb{E}\bigg[r_t^2\bigg] &= \mathbb{E}\bigg[\mu_t^2 + 2\,\mu_t\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\}\,\sigma\int\limits_s^t \mathrm{Exp}\bigg\{\lambda\,u\bigg\}\,\delta W_u + \mathrm{Exp}\bigg\{-2\,\lambda\,t\bigg\}\,\sigma^2\int\limits_s^t\int\limits_s^t \mathrm{Exp}\bigg\{\lambda\,(u+v)\bigg\}\,\delta W_u\,\delta W_v\bigg]\\
&= \mu_t^2 + 2\,\mu_t\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\}\,\sigma\int\limits_s^t \mathrm{Exp}\bigg\{\lambda\,u\bigg\}\,\mathbb{E}\bigg[\delta W_u\bigg] + \mathrm{Exp}\bigg\{-2\,\lambda\,t\bigg\}\,\sigma^2\,\mathbb{E}\bigg[\int\limits_s^t\int\limits_s^t \mathrm{Exp}\bigg\{\lambda\,(u+v)\bigg\}\,\delta W_u\,\delta W_v\bigg]\\
&= \mu_t^2 + \frac{1}{2}\,\sigma^2\bigg(1-\mathrm{Exp}\bigg\{-2\,\lambda\,(t-s)\bigg\}\bigg)\lambda^{-1} \qquad (16)\end{aligned}$$

Using Equation (14) above the mean of the short rate at time t given the short rate at time s is...

$$\mathrm{mean} = \mathbb{E}\bigg[r_t\bigg] = r_\infty + \mathrm{Exp}\bigg\{-\lambda\,(t-s)\bigg\}\bigg(r_s - r_\infty\bigg) \qquad (17)$$

Using Equations (12), (16) and (17) above the variance of the short rate at time t given the short rate at time s is...

$$\begin{aligned}\mathrm{variance} &= \mathbb{E}\bigg[r_t^2\bigg] - \bigg(\mathbb{E}\bigg[r_t\bigg]\bigg)^2\\
&= \mu_t^2 + \frac{1}{2}\,\sigma^2\bigg(1-\mathrm{Exp}\bigg\{-2\,\lambda\,(t-s)\bigg\}\bigg)\lambda^{-1} - \mu_t^2\\
&= \frac{1}{2}\,\sigma^2\bigg(1-\mathrm{Exp}\bigg\{-2\,\lambda\,(t-s)\bigg\}\bigg)\lambda^{-1} \qquad (18)\end{aligned}$$

Short Rate Covariance

Imagine that we are currently sitting at time s and that there are two future time periods t and u. Given that r_s is the short rate at time s (known) the equation for the short rates at time t (random) and at time u (random) can be written as...

$$r_t = r_s + \int\limits_s^t \delta r_x \ \text{...and...}\ r_u = r_s + \int\limits_s^u \delta r_y \qquad (19)$$

Because the increments in the driving Brownian motion over the time interval $[s, min(t,u)]$ will be the same for short rates r_t and r_u and therefore the two short rates are positively correlated. The equation for the covariance between the short rate at time t and the short rate at time u is...

$$\mathrm{Covariance}\bigg[r_t\,r_u\bigg] = \mathbb{E}\bigg[r_t\,r_u\bigg] - \mathbb{E}\bigg[r_t\bigg]\mathbb{E}\bigg[r_u\bigg] \qquad (20)$$

Using Equation (13) above we can write the equations for the short rate at time t and at time u as...

$$r_t = \mu_t + \mathrm{Exp}\bigg\{-\lambda\,t\bigg\}\,\sigma\int\limits_s^t \mathrm{Exp}\bigg\{\lambda\,x\bigg\}\,\delta W_x \ \text{...and...}\ r_u = \mu_u + \mathrm{Exp}\bigg\{-\lambda\,u\bigg\}\,\sigma\int\limits_s^u \mathrm{Exp}\bigg\{\lambda\,y\bigg\}\,\delta W_y \qquad (21)$$

Per covariance Equation (20) above we need the expectation of the product of the two short rates r_t and r_u. Using Equation (21) above this expectation in equation form is...

$$\mathbb{E}\bigg[r_t\,r_u\bigg] = \mathbb{E}\bigg[\bigg(\mu_t + \mathrm{Exp}\bigg\{-\lambda\,t\bigg\}\,\sigma\int\limits_s^t \mathrm{Exp}\bigg\{\lambda\,x\bigg\}\,\delta W_x\bigg)\bigg(\mu_u + \mathrm{Exp}\bigg\{-\lambda\,u\bigg\}\,\sigma\int\limits_s^u \mathrm{Exp}\bigg\{\lambda\,y\bigg\}\,\delta W_y\bigg)\bigg] \qquad (22)$$

Given that the expected value of the product of δW_x and δW_y is zero (see Appendix Equation (38) below) we can ignore those cross products and rewrite Equation (22) above as...

$$\mathbb{E}\bigg[r_t\,r_u\bigg] = \mu_t\,\mu_u + \sigma^2\,\mathrm{Exp}\bigg\{-\lambda\,(t+u)\bigg\}\,\mathbb{E}\bigg[\int\limits_s^t\int\limits_s^u \mathrm{Exp}\bigg\{\lambda\,(x+y)\bigg\}\,\delta W_x\delta W_y\bigg] \qquad (23)$$

Because the cross product δW_x and δW_y is equal to zero we can rewrite Equation (23) above as (note that $t \wedge u$ means that you take the minimum of t and u)...

$$\mathbb{E}\bigg[r_t\, r_u\bigg] = \mu_t\, \mu_u + \sigma^2\, \mathrm{Exp}\bigg\{-\lambda\,(t+u)\bigg\}\,\mathbb{E}\bigg[\int\limits_s^{t\wedge u}\int\limits_s^{t\wedge u} \mathrm{Exp}\bigg\{\lambda\,(x+y)\bigg\}\,\delta W_x \delta W_y\bigg] \tag{24}$$

Using Appendix Equation (47) below the solution to Equation (24) above is...

$$\mathbb{E}\bigg[r_t\, r_u\bigg] = \mu_t\, \mu_u + \frac{1}{2}\,\sigma^2\, \mathrm{Exp}\bigg\{-\lambda\,(t+u)\bigg\}\bigg(\mathrm{Exp}\bigg\{2\,\lambda\,(t\wedge u)\bigg\} - \mathrm{Exp}\bigg\{2\,\lambda\, s\bigg\}\bigg)\lambda^{-1} \tag{25}$$

Using Equations (14) and (25) above we can write the covariance of the random short rates at time t and u given the known short rate at time s as...

$$\begin{aligned}
\mathrm{Covariance}\,(r_t\, r_u) &= \mathbb{E}\bigg[r_t\, r_u\bigg] - \mathbb{E}\bigg[r_t\bigg]\mathbb{E}\bigg[r_u\bigg] \\
&= \mu_t\, \mu_u + \frac{1}{2}\,\sigma^2\, \mathrm{Exp}\bigg\{-\lambda\,(t+u)\bigg\}\bigg(\mathrm{Exp}\bigg\{2\,\lambda\,(t\wedge u)\bigg\} - \mathrm{Exp}\bigg\{2\,\lambda\, s\bigg\}\bigg)\lambda^{-1} - \mu_t\, \mu_u \\
&= \frac{1}{2}\,\sigma^2\, \mathrm{Exp}\bigg\{-\lambda\,(t+u)\bigg\}\bigg(\mathrm{Exp}\bigg\{2\,\lambda\,(t\wedge u)\bigg\} - \mathrm{Exp}\bigg\{2\,\lambda\, s\bigg\}\bigg)\lambda^{-1} \qquad (26)
\end{aligned}$$

Simulating The Short Rate

Given that r_t is the normally-distributed short rate at time t, m is the short rate **mean** per Equation (17) above, and v is the short rate **variance** per Equation (18) above, we can define the normalized random variable Z to be the following equation...

$$\frac{r_t - m}{\sqrt{v}} = Z \;\; \text{...such that...} \;\; Z \sim N\bigg[0, 1\bigg] \tag{27}$$

By rearranging Equation (27) above the simulated value of the short rate at some future time t given the short rate at time $s < t$ is...

$$r_t = m + \sqrt{v}\, Z \;\; \text{...where...} \;\; Z \sim N\bigg[0, 1\bigg] \tag{28}$$

Note that one of the potential issues with the Vasicek interest rate process is that you can get negative rates. If we define the function CNDF(value) to be the cumuative distribution function of a normally-distributed random variable with mean zero and variance and use Equation (28) above then the probability of pulling a negative rate from the rate distribution is...

$$\mathrm{Prob}\bigg[r_t < 0\bigg] = \mathrm{CNDF}(Z) \;\; \text{...because...} \;\; \text{if} \;\; r_t = 0 \;\; \text{then} \;\; Z = -\frac{\text{mean}}{\sqrt{\text{variance}}} \tag{29}$$

When Short-Term Rate Is Random

In the equations above the equation for the random short rate at time t (see Equation (11) above) assumes that the short-rate at time s was known (i.e. not random). What happens to that equation when from the perspective of time zero the short-rate at time $s > 0$ is not known and therefore is a random variable? Using Equation (11) above the equation for the short rate at time t ***given*** the short rate at time s is...

$$r_t = r_\infty + \mathrm{Exp}\bigg\{-\lambda\,(t-s)\bigg\}(r_s - r_\infty) + \mathrm{Exp}\bigg\{-\lambda\, t\bigg\}\,\sigma\int\limits_s^t \mathrm{Exp}\bigg\{\lambda\, u\bigg\}\,\delta W_u \tag{30}$$

If the short rate r_s in Equation (30) above is a random variable (***is not a given***) then the equation for the random short rate at time s given the known short-rate at time zero is...

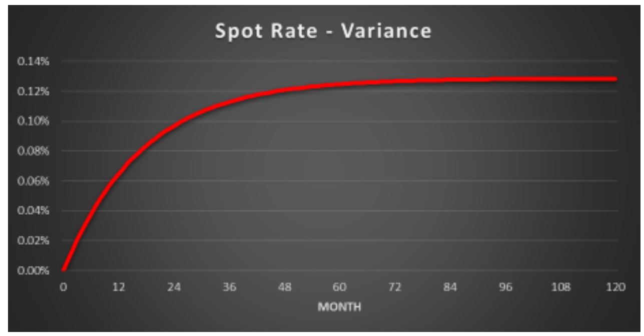
$$r_s = r_\infty + \mathrm{Exp}\bigg\{-\lambda\, s\bigg\}(r_0 - r_\infty) + \mathrm{Exp}\bigg\{-\lambda\, s\bigg\}\,\sigma\int\limits_0^s \mathrm{Exp}\bigg\{\lambda\, v\bigg\}\,\delta W_v \tag{31}$$

Using Appendix Equation (48) below the equation for the random short rate at time t (Equation (30) above) ***given the random*** short rate at time s (Equation (31) above) is...

$$= r_\infty + \mathrm{Exp}\left\{ -\lambda\, t \right\} (r_0 - r_\infty) + \mathrm{Exp}\left\{ -\lambda\, t \right\} \sigma \int_0^t \mathrm{Exp}\left\{ \lambda\, w \right\} \delta W_w \tag{32}$$

Answers To Our Hypothetical Problem

Question 1: Graph the short rate curve (mean and variance) over the time interval $[0, 10]$ - Uses Equations (17) and (18) above.



Question 2: What is the short rate mean and variance at the end of years 1 and 3? - Uses Equations (17) and (18) above.

The expected short rate at the end of year 1 is...

$$\text{mean} = 0.09 + \mathrm{Exp}\left\{ -0.35 \times (1-0) \right\} \left(0.04 - 0.09 \right) = 5.477\%$$
$$\text{variance} = \frac{1}{2} \times 0.03^2 \times \left(1 - \mathrm{Exp}\left\{ -2 \times 0.35 \times (1-0) \right\} \right) \times 0.35^{-1} = 0.065\% \tag{33}$$

The expected short rate at the end of year 3 is...

$$\text{mean} = 0.09 + \mathrm{Exp}\left\{ -0.35 \times (3-0) \right\} \left(0.04 - 0.09 \right) = 7.250\%$$
$$\text{variance} = \frac{1}{2} \times 0.03^2 \times \left(1 - \mathrm{Exp}\left\{ -2 \times 0.35 \times (3-0) \right\} \right) \times 0.35^{-1} = 0.113\% \tag{34}$$

Question 3: What is the correlation between the random short rates at the end of years 1 and 3?

The covariance between the short rates at the end of year 1 and year 3 is... (uses Equation (26) above)

$$\begin{aligned} \mathrm{Cov}\,(r_1\, r_3) &= \frac{1}{2} \times 0.03^2 \times \mathrm{Exp}\left\{ -0.35 \times (1+3) \right\} \times \left(\mathrm{Exp}\left\{ 2 \times 0.35 \times (1 \wedge 3) \right\} - \mathrm{Exp}\left\{ 2 \times 0.35 \times 0 \right\} \right) \times 0.35^{-1} \\ &= 0.00032 \end{aligned} \tag{35}$$

Using the variance of the short rate at the end of years 1 and 3 as calculated in Question 2 above, the correlation between the random short rate at the end of year 1 and year 3 is...

$$\mathrm{Correl}\ (r_1\, r_3) = \frac{\mathrm{CoVar}\,(r_1 r_3)}{\mathrm{Sdev}\ r_1 \times \mathrm{Sdev}\ r_3} = \frac{0.00032}{\sqrt{0.00065} \times \sqrt{0.00113}} = 0.38 \tag{36}$$

Question 4: What is the probability that the random short rate at the end of year 3 will be negative?

Using Equations (29) and (34) above the probability of a negative rate is...

$$\mathrm{Prob}\left[r_3 < 0 \right] = \mathrm{CNDF}\left(-\frac{\text{mean}}{\sqrt{\text{variance}}} \right) = \mathrm{CNDF}\left(-\frac{0.07250}{\sqrt{0.00113}} \right) = 1.55\% \tag{37}$$

Appendix

A. Note the following expectations applicable to the change in the Brownian motion W_u over the infinitesimally small time interval $[u, u+\delta u]$...

$$\mathbb{E}\Big[\delta W_u\Big] = 0 \text{ ...and... } \mathbb{E}\Big[\delta W_u^2\Big] = \delta u \text{ ...and... } \mathbb{E}\Big[\delta W_u\,\delta W_v\Big] = 0 \tag{38}$$

B. We want to find the expected value of the following equation...

$$\mathbb{E}\bigg[\bigg(\int\limits_s^t \mathrm{Exp}\Big\{\lambda\,u\Big\}\,\delta W_u\bigg)^2\bigg] = \mathbb{E}\bigg[\int\limits_s^t\int\limits_s^t \mathrm{Exp}\Big\{\lambda\,u\Big\}\,\delta W_u\,\mathrm{Exp}\Big\{\lambda\,v\Big\}\,\delta W_v\bigg] = \mathbb{E}\bigg[\int\limits_s^t\int\limits_s^t \mathrm{Exp}\Big\{\lambda\,(u+v)\Big\}\,\delta W_u\,\delta W_v\bigg] \tag{39}$$

Per Appendix Equation (38) above, the product of δW_u and δW_v is zero when $u \neq v$ and therefore we need only consider cases where $u = v$. We can rewrite Appendix Equation (39) above as...

$$\mathbb{E}\bigg[\bigg(\int\limits_s^t \mathrm{Exp}\Big\{\lambda\,u\Big\}\,\delta W_u\bigg)^2\bigg] = \mathbb{E}\bigg[\int\limits_s^t \mathrm{Exp}\Big\{\lambda\,2\,u\Big\}\,\delta W_u^2\bigg] \tag{40}$$

Per Appendix Equation (38) above, the square of $\delta W_u = \delta u$. We can rewrite Appendix Equation (40) above as...

$$\mathbb{E}\bigg[\bigg(\int\limits_s^t \mathrm{Exp}\Big\{\lambda\,u\Big\}\,\delta W_u\bigg)^2\bigg] = \int\limits_s^t \mathrm{Exp}\Big\{\lambda\,2\,u\Big\}\,\mathbb{E}\Big[\delta W_u^2\Big] = \int\limits_s^t \mathrm{Exp}\Big\{\lambda\,2\,u\Big\}\,\delta u \tag{41}$$

The solution to Equation (41) is...

$$\mathbb{E}\bigg[\bigg(\int\limits_s^t \mathrm{Exp}\Big\{\lambda\,u\Big\}\,\delta W_u\bigg)^2\bigg] = \frac{1}{2}\bigg(\mathrm{Exp}\Big\{2\,\lambda\,t\Big\} - \mathrm{Exp}\Big\{2\,\lambda\,s\Big\}\bigg)\lambda^{-1} \tag{42}$$

C. We want to solve the following equation...

$$I = \mathrm{Exp}\Big\{-2\,\lambda\,t\Big\}\,\sigma^2\,\mathbb{E}\bigg[\int\limits_s^t\int\limits_s^t \mathrm{Exp}\Big\{\lambda\,(u+v)\Big\}\,\delta W_u\,\delta W_v\bigg] \tag{43}$$

Using Appendix Equation (42) above we can rewrite Equation (43) above as...

$$I = \frac{1}{2}\,\sigma^2\,\mathrm{Exp}\Big\{-2\,\lambda\,t\Big\}\bigg(\mathrm{Exp}\Big\{2\,\lambda\,t\Big\} - \mathrm{Exp}\Big\{2\,\lambda\,s\Big\}\bigg)\lambda^{-1} \tag{44}$$

The solution to Equation (44) is...

$$I = \frac{1}{2}\,\sigma^2\bigg(1 - \mathrm{Exp}\Big\{-2\,\lambda\,(t-s)\Big\}\bigg)\lambda^{-1} \tag{45}$$

D. We want to solve the following equation...

$$I = \mathrm{Exp}\Big\{-\lambda\,(y+z)\Big\}\,\sigma^2\,\mathbb{E}\bigg[\int\limits_x^{y\wedge z}\int\limits_x^{z\wedge y} \mathrm{Exp}\Big\{\lambda\,(u+v)\Big\}\,\delta W_u\,\delta W_v\bigg] \tag{46}$$

Using Appendix Equation (42) above we can rewrite Equation (43) above as...

$$I = \frac{1}{2}\,\sigma^2\,\mathrm{Exp}\Big\{-\lambda\,(y+z)\Big\}\bigg(\mathrm{Exp}\Big\{2\,\lambda\,(y\wedge z)\Big\} - \mathrm{Exp}\Big\{2\,\lambda\,x\Big\}\bigg)\lambda^{-1} \tag{47}$$

E. After substituting the random variable r_s in Equation (31) into short rate Equation (30) above that equation becomes...

$$r_t = r_\infty + \mathrm{Exp}\left\{ -\lambda\,(t-s)\right\}\left(r_\infty + \mathrm{Exp}\left\{ -\lambda\,s\right\}(r_0 - r_\infty) + \mathrm{Exp}\left\{ -\lambda\,s\right\}\sigma \int_0^s \mathrm{Exp}\left\{\lambda\,v\right\}\delta W_v - r_\infty\right)$$

$$+ \mathrm{Exp}\left\{ -\lambda\,t\right\}\sigma \int_s^t \mathrm{Exp}\left\{\lambda\,u\right\}\delta W_u$$

$$= r_\infty + \mathrm{Exp}\left\{ -\lambda\,t\right\}\mathrm{Exp}\left\{\lambda\,s\right\}\left(\mathrm{Exp}\left\{ -\lambda\,s\right\}(r_0 - r_\infty) + \mathrm{Exp}\left\{ -\lambda\,s\right\}\sigma \int_0^s \mathrm{Exp}\left\{\lambda\,v\right\}\delta W_v\right)$$

$$+ \mathrm{Exp}\left\{ -\lambda\,t\right\}\sigma \int_s^t \mathrm{Exp}\left\{\lambda\,u\right\}\delta W_u$$

$$= r_\infty + \mathrm{Exp}\left\{ -\lambda\,t\right\}(r_0 - r_\infty) + \mathrm{Exp}\left\{ -\lambda\,t\right\}\sigma \int_0^s \mathrm{Exp}\left\{\lambda\,v\right\}\delta W_v\Big) + \mathrm{Exp}\left\{ -\lambda\,t\right\}\sigma \int_s^t \mathrm{Exp}\left\{\lambda\,u\right\}\delta W_u$$

$$= r_\infty + \mathrm{Exp}\left\{ -\lambda\,t\right\}(r_0 - r_\infty) + \mathrm{Exp}\left\{ -\lambda\,t\right\}\sigma \int_0^t \mathrm{Exp}\left\{\lambda\,w\right\}\delta W_w \tag{48}$$